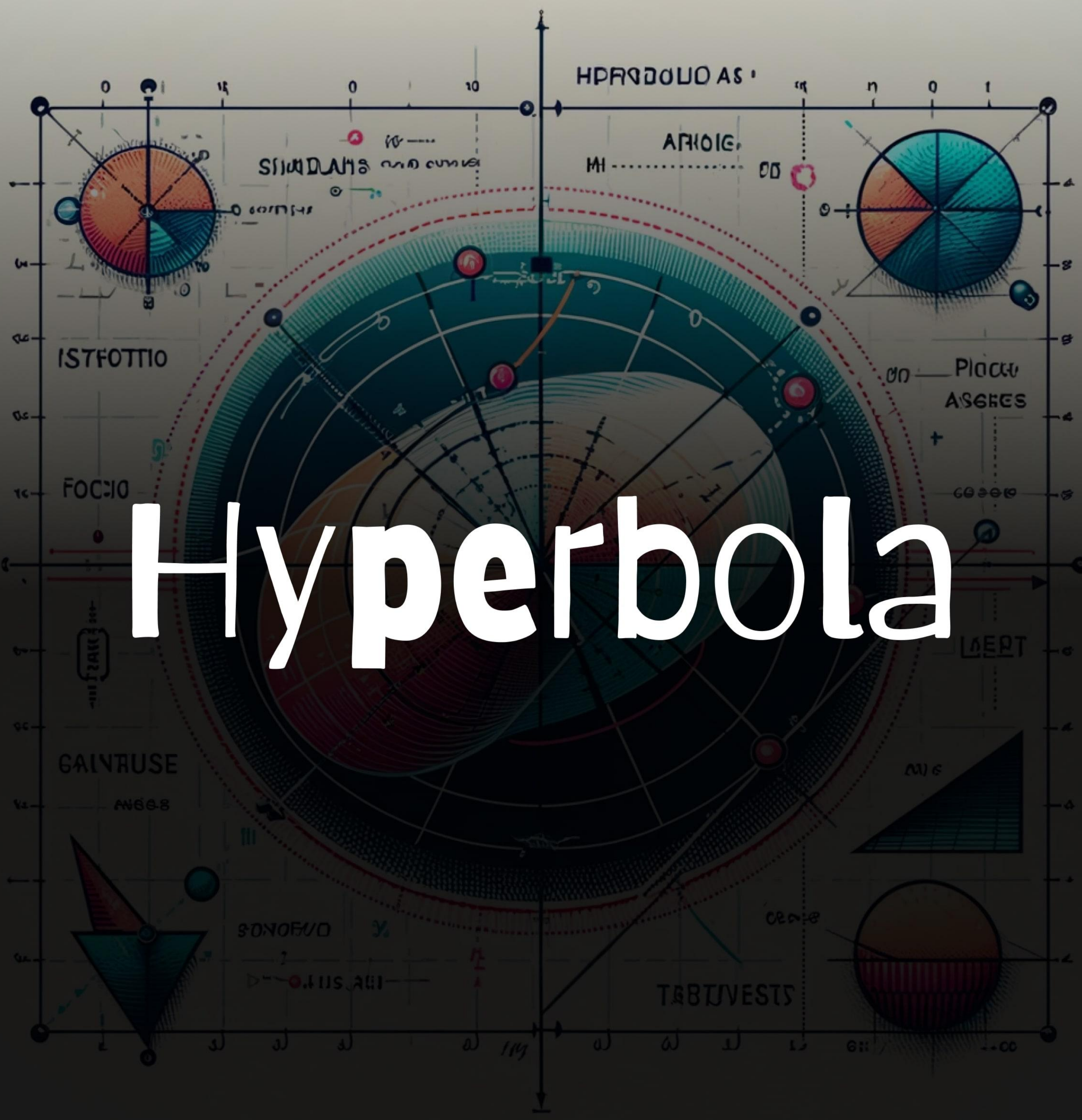
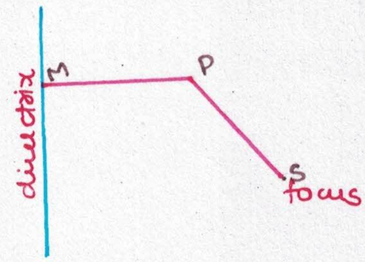
Hyperbola



HYPERBOLA

DEFINATION The locus of the pt. which moves in such a way that the ratio of its distance from a fixed pt. and a fixed line is always contant and greater than 1.

i.e. $\frac{SP}{PM} = e$ $\qquad e > 1$



Ques: Find the eq. of hyperbola whose focus is (4,0) and directrix is $x = 1$ and $e = 2$

Sol: $(x-4)^2 + y^2 = 4(x-1)^2$

$\Rightarrow x^2 + y^2 - 8x + 16 = 4x^2 + 4 - 8x$

$\Rightarrow -3x^2 - y^2 - 12 = 0$

$\Rightarrow 3x^2 - y^2 = 12$

$\Rightarrow \boxed{\frac{x^4}{4} - \frac{y^2}{12} = 1}$

Ques: $S(ae, 0)$ directrix : $x = \frac{a}{e}$ ecc. $= e$

Sol: $(x-ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$

$\Rightarrow x^2 + a^2e^2 + y^2 - 2aex = e^2\left(x^2 + \frac{a^2}{e^2} - \frac{2a}{e}x\right)$

$\Rightarrow x^2 + y^2 - 2ae\,x + a^2e^2 = e^2x^2 + a^2 - 2aex$

$\Rightarrow x^2(e^2-1) - y^2 + a^2(1-e^2) = 0$

$\Rightarrow x^2(e^2-1) - y^2 = a^2(e^2-1)$

$\Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1}$

$\Rightarrow a^2(e^2-1) = b^2$

$\Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$

$a^2(e^2-1) = b^2$

$e^2 - 1 = \frac{b^2}{a^2} \Rightarrow e^2 = \frac{b^2}{a} + 1$

$$\boxed{e = \sqrt{\frac{a^2+b^2}{a^2}}}$$

The 2nd degree general equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents hyperbola if

$\Delta \neq 0 \Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$

TERMINOLOGY

1. **Transverse Axis (TA):** The biggest line of symmetry which contains foci as well as vertex of the hyperbola is known as TA

2. **Conjugate Axis (CA):** The line perpendicular to transverse axis and passing through the centre of the hyperbola is known as CA

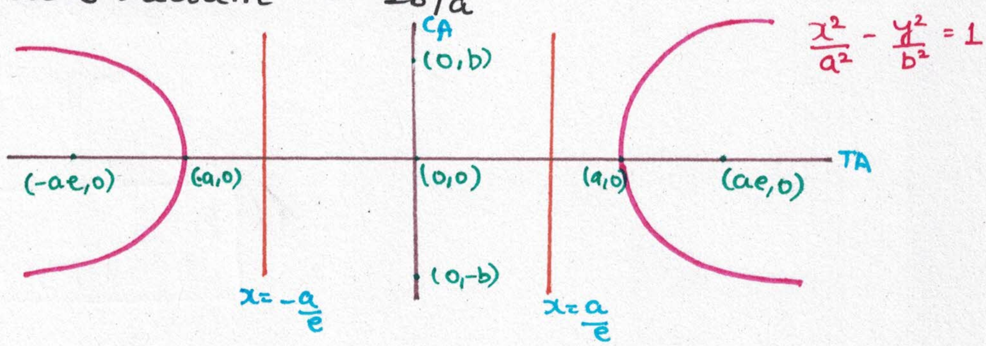
3. **Centre:** The POI of TA and CA is known as the centre of the hyperbola which is also the mid pt. of 2 foci and 2 vertex

4. Here keep in mind that the centre of the hyperbola lies outside.

Length of Transverse axis = $2a$

Length of Conjugate axis = $2b$

Length of latus rectum = $2b^2/a$



$C = (0,0)$; $V = (\pm a, 0)$; $S = (\pm ae, 0)$

End pts of CA $= (0, \pm b)$

Eq. of TA $\quad y = 0$

Eq. of CA $\quad x = 0$

Eq^n of directrix :- $x = \pm \frac{a}{e}$

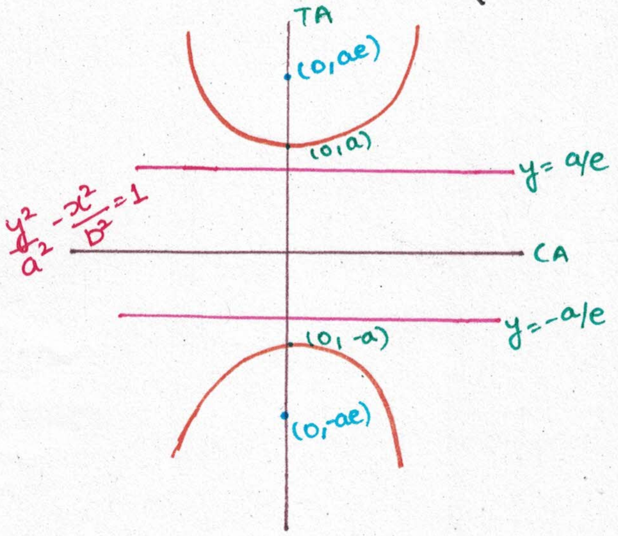
Eq^n of LR $x = \pm ae$

LOTA $= 2a$

LOLA $= 2b$

LOLR $= \frac{2b^2}{a}$

End pts of LR $= \left(\pm ae, \pm \frac{b^2}{a}\right)$



$C = (0,0)$ $V = (0, \pm a)$

$S = (0, \pm ae)$

End pts. CA $= (\pm b, 0)$

Eq^n of TA $x = 0$

Eq^n of CA $y = 0$

Eq^n of directrix $y = \pm \frac{a}{e}$

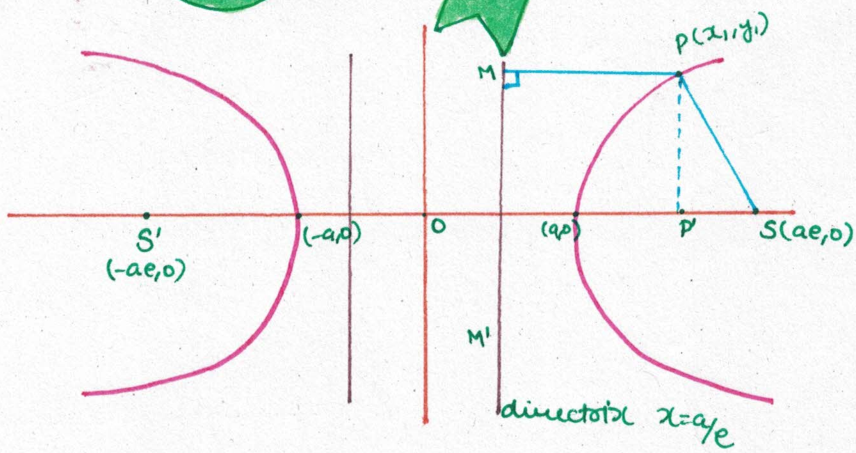
Eq^n of LR $y = \pm ae$

LOTA $= 2a$ LOCA $= 2b$

LOLR $= \frac{2b^2}{a}$

End pts. of LR $\left(\pm \frac{b^2}{a}, \pm ae\right)$

FOCAL LENGTH



$PS = e\,PM$ $PS = e\,P'M'$

$PS = e(OP' - OM')$

$PS = e\left(|x_1| - \frac{a}{e}\right)$

$\boxed{PS = e|x_1| - a}$

$PS' = ePN$ $PS' = eP'N'$

$PS' = e(OP' + ON')$

$PS' = e\left(|x| + \frac{a}{e}\right)$

$\boxed{PS' = e|x_1| + a}$

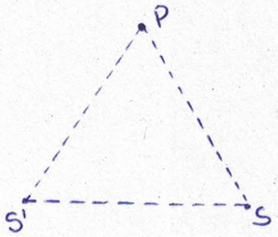
$\boxed{|PS' - PS| = 2a}$

* The difference of focal lengths of a points from the foci is always constant and equal to 2a.

ALTERNATIVE DEFINATION OF HYPERBOLA

The locus of a pt. P which moves in such a way that the difference of the distance from two fixed points is always constant and less than the distance between two fixed pts.

NOTE

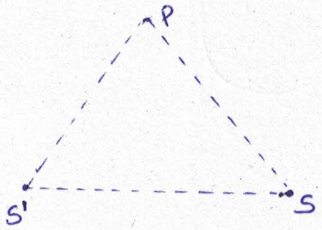


$PS + PS' > SS'$ Ellipse

$PS + PS' = SS'$ Line segment

$PS + PS' < SS'$ No such pt. possible

NOTE



$|PS - PS'| < SS'$ Hyperbola

$|PS - PS'| = SS'$ A pair of ray

$|PS - PS'| > SS'$ No pt. possible

Ques: Find the locus of the pt. which moves in such a way that it difference from 2 fixed pts. (3,0) and (-3,0) is 4.

Sol: $|PS - PS'| = 4 = 2a$, $a = 2$

$(ae, 0) = (3, 0)$ $2e = 3$ $e = 3/2$

$\frac{a}{4} = \frac{4 + b^2}{4}$ $b^2 = 5$ $b = \sqrt{5}$

$\frac{x^2}{4} - \frac{y^2}{5} = 1$

Position of a point w.r.t. Hyperbola

If we wish to find the position of a pt. $P(x_1, y_1)$ w.r.t. hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we do following steps-

I STEP Find the center of the hyperbola and substitute it in the equation to obtain the sign of output.

In case hyperbola is not given in the standard form, find the center of the hyperbola using partial differentiation and then substitute the pt. in the equation.

II STEP The sign of output which we have obtained in Step I is preserved for all the pts. lying outside the hyperbola.

III STEP Now substitute the pt. $P(x_1, y_1)$ and note the sign of output.

If sign of output is same as in step I then pt. is lying outside. If opposite of step I then pt. is inside. and if equal to zero, then pt. is on $\mathcal{H}$

Ques: Find the position of pt. (0, 1) w.r.t. hyperbola $\frac{x^2}{5} - \frac{y^2}{9} = 1$

Sol: $9x^2 - 5y^2 = 45$

$0 - 1 - 45$ Inside

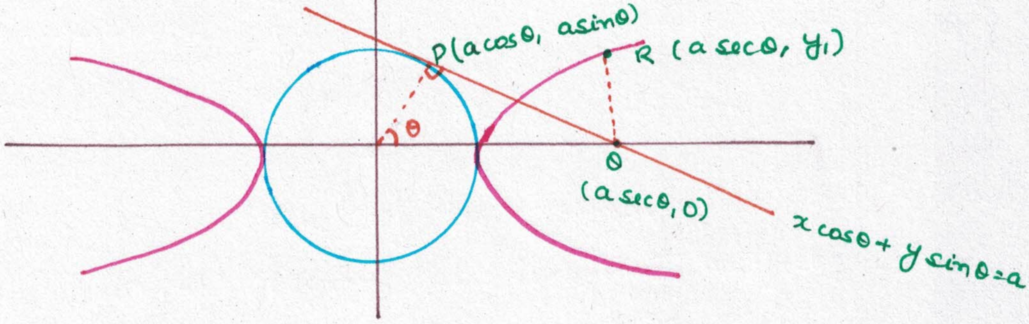
$-5(1)^2 - 45 = 0$

AUXILIARY CIRCLE

The locus of foot of the perpendicular drawn from foci to any tangent is called auxiliary circle.

In case of hyperbola, the equation of auxiliary circle is-

$$x^2 + y^2 = a^2$$



$R = (a \sec\theta, y_1)$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{a^2 \sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1$

$y^2/b^2 = \sec^2\theta - 1$

$y^2 = b^2 \tan^2\theta \Rightarrow y = b\tan\theta$

$\boxed{R = (a \sec\theta, b\tan\theta)}$ $\theta \in [0, 2\pi] - \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

Parametric coordinates

TANGENT AND NORMAL AT THE POINT

POINT FORM

$T = 0$

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

PARAMETRIC FORM

$\frac{x(a\sec\theta)}{a^2} - \frac{y(b\tan\theta)}{b^2} = 1$

$\frac{x \sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

SLOPE FORM

$y = mx + c$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Ellipse equation : $y = mx \pm \sqrt{a^2m^2 + b^2}$

Hyperbola :- $y = mx \pm \sqrt{a^2m^2 - b^2}$

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$x^2 - a^2m^2x^2 - a^2c^2 - 2a^2mxc = a^2b^2$$

$$x^2(1 - a^2m^2) - 2a^2mxc - a^2c^2 - a^2b^2 = 0$$

$D = 0$

$c = \pm\sqrt{a^2m^2 - b^2}$

Equation:

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Point of contact: $\left(\frac{\pm a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2-b^2}}\right)$

Normal:

<1> Point form:

$m_T = \frac{x_1 b^2}{a^2 y_1}$ $\quad m_n = -\frac{a^2 y_1}{b^2 x_1}$ $\quad$ pt. (x_1, y_1)

$$(y - y_1) = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

<2> Parametric Form:

$m_T = \frac{b\sec\theta}{a\tan\theta}$ $\quad m_n = -\frac{a\tan\theta}{b\sec\theta}$ $\quad m_n = -\frac{a\sin\theta}{b}$

Case-I If in slope form of tangent $\{c = \sqrt{a^2m^2-b^2}$

$a^2m^2 - b^2 > 0$

$m \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$

If slope of the tangent lies in the above interval then, we can draw 2 distinct tangents which touches the different branches.

Case-II $a^2m^2 - b^2 = 0$

$m = -\frac{b}{a}$ or $m = \frac{b}{a}$

In this case, the two tangents will always pass through the centre of the hyperbola and these tangents are supposed to touch the graph at ∞ and such type of tangents called asymptotes.

Case-III $a^2m^2 - b^2 < 0$

$$m \in \left(-\frac{b}{a}, \frac{b}{a}\right)$$

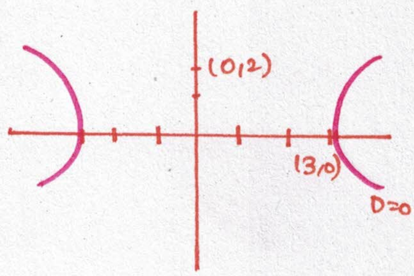
If the slope lies in above interval then, we cannot draw any tangent.

☆ The point of contanct of tangent and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ will be given as-

$$\left(\frac{\mp a^2 m}{\sqrt{a^2m^2 - b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2-b^2}}\right)$$

Ques: Find the equation of tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, with slope $= 1$. and also check the obtained tangent is touching the curve again or not.

Sol: $\frac{b}{a} = \frac{2}{3}$ $\quad m = 1 \quad m > \frac{b}{a}$

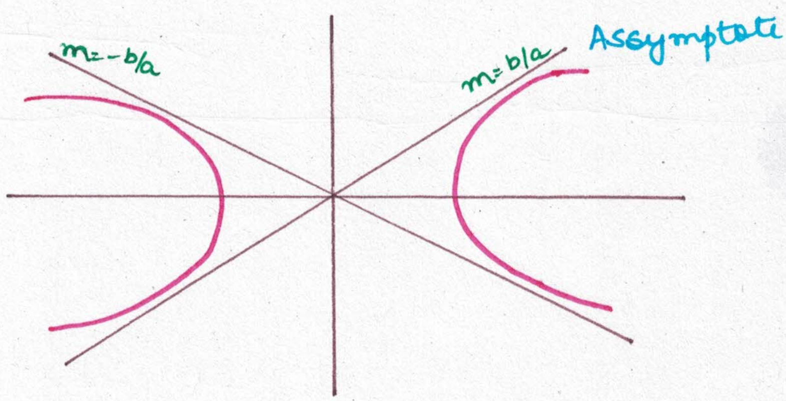


$$y = x \pm \sqrt{9(1) - 4}$$

$$y = x \pm \sqrt{5}$$ 2 parallel tangents

The tangent will touch the curve at only 1 pt.

$$POC = \mp \frac{9}{\sqrt{5}}, \mp \frac{4}{\sqrt{5}}$$



EQUATION OF NORMAL

<1> POINT FORM:

Ellipse- $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

Hyperbola - $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

<2> PARAMETRIC FORM:

$$\frac{a^2 x}{a\sec\theta} + \frac{b^2 y}{b\tan\theta} = a^2+b^2$$

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2$$

Ques: Find the area bounded by tangent to the curve $\frac{x^2}{16} - \frac{y^2}{9} = 1$ with TA and CA in slope form.

Sol: $y = mx + \sqrt{16m^2-9}$

TA : $y=0$ $\quad x = \frac{\sqrt{16m^2-9}}{m}$

$y = \sqrt{16m^2-9}$

$$\text{area} = \frac{1}{2}\left[\frac{\sqrt{16m^2-9}}{m} \times \sqrt{16m^2-9}\right] = \frac{1}{2}\frac{(16m^2-9)}{m}$$

EQUATION OF TANGENT FROM THE POINT

COMBINED FORM

If pt. $P(x_1, y_1)$ lies outside the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then we can draw 2 tangents to the hyperbola whose combined eq. is given as-

$T^2 = SS_1$

$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \qquad S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Ques: Find the eq. of tangent of hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ from the pt. $(3,0)$.

Sol: $T^2 = S_1$

$$\Rightarrow \left(\frac{3x}{16} - 0 - 1\right)^2 = \left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right)\left(\frac{9}{16} - 1\right)$$

$$\Rightarrow \left(\frac{3x-16}{16}\right)^2 = \left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right)\left(-\frac{7}{16}\right)$$

$$\Rightarrow \frac{9x^2 + 256 - 96x}{256} = -\frac{7x^2}{256} + \frac{7y^2}{144} + \frac{7}{16}$$

$$\Rightarrow 9x^2 + 256 - 96x = +112 - 7x^2 + 63y^2$$

$\Rightarrow \quad 16x^2 - 63y^2 - 96x + 144 = 0$

$\Rightarrow \quad 9x^2 - 7y^2 - 54x + 81 = 0$

$\Rightarrow \quad (3x + \sqrt{7}y)(3x - \sqrt{7}y) = 0$

$\Rightarrow \quad (3x + \sqrt{7}y + \alpha)(3x - \sqrt{7}y + \beta) = 0$

$\Rightarrow 9x^2 - 7y^2 - 3\sqrt{7}xy + 3x\beta + 3\sqrt{7}xy + \sqrt{7}\beta y + 3\alpha x - \sqrt{7}\alpha y + \alpha\beta = 0$

$\Rightarrow \quad 9x^2 - 7y^2 + x(3\beta + 3\alpha) + y(\sqrt{7}\beta - \sqrt{7}\alpha) + \alpha\beta = 0$

$\alpha + \beta = -18 \qquad\qquad \alpha = \beta$

$2\beta = -18 \qquad\qquad \alpha, \beta = -9$

$3x + \sqrt{7}y - 9 = 0 \quad , \quad 3x - \sqrt{7}y - 9 = 0$

Separate Form If we wish to find the equation of tangent in separate Form (Tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$)

Step 1: Write the equation of tangent in slope form

Step 2: Substitute the pt. $P(x_1, y_1)$ to form a quardratic eq. in 'm'.

Step 3: We will get 2 values of m and 4 equation of lines out of which only 2 lines will pass from (x_1, y_1)

Ques: Find the eq. of tangent from pt. $(3,0)$ to $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Sol:

$y = mx \pm \sqrt{a^2m^2 - b^2}$

$0 = 3m + \sqrt{16m^2 - 9}$

$9m^2 = 16m^2 - 9$

$9 = 7m^2$

$m = \pm \frac{3}{\sqrt{7}}$

$\sqrt{a^2m^2 - b^2} = \sqrt{16\left(\frac{9}{7}\right) - 9} = \frac{9}{\sqrt{7}}$

$(3,0) \quad y = \frac{3}{\sqrt{7}}x + \frac{9}{\sqrt{7}}$ ✗ $\qquad \boxed{y = \frac{3x}{\sqrt{7}} - \frac{9}{\sqrt{7}}}$

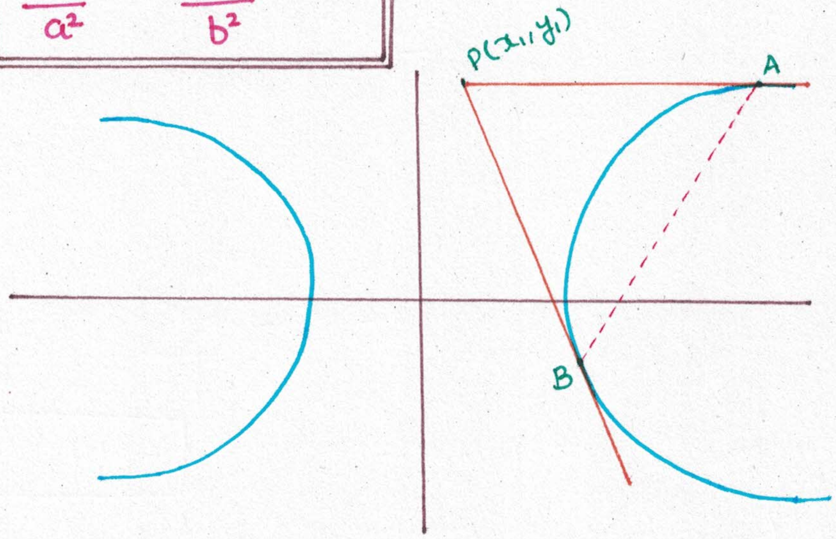
$\boxed{y = \frac{-3x}{\sqrt{7}} + \frac{9}{\sqrt{7}}} \qquad y = \frac{-3x}{\sqrt{7}} - \frac{9}{\sqrt{7}}$ ✗

CHORD OF CONTACT

If pt. $P(x_1, y_1)$ lies outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then we can draw 2 tangents from this point. The line joining the pt. of contact is known as chord of contact.

Equation :- $T = 0$

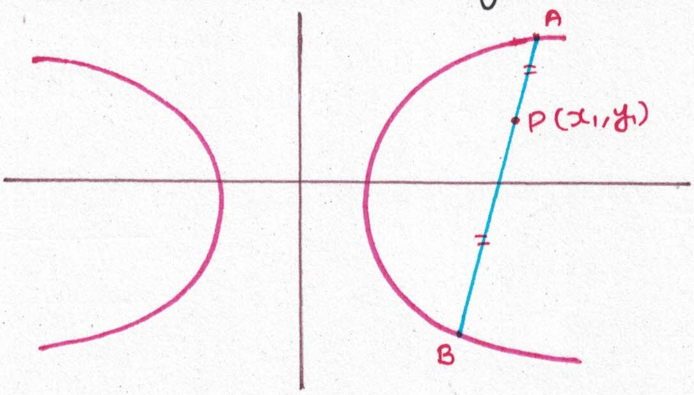
$$\boxed{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1}$$



$AB \Rightarrow T = 0$

Chord whose mid pt. is given.

If pt. $P(x_1, y_1)$ is a mid pt. of the chord of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the eq. of this chord will be given as $\boxed{T = S_1}$



NOTE The locus of mid pt. of parallel chords of hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called diameter of hyperbola and its eq is-

$$\boxed{Y = \frac{b^2 x}{a^2 m}}$$

$m \rightarrow$ slope of chord

Ques: If we join the 2 pts. of hyperbola whose parametric co-ordinates are : A $(a \sec\alpha, b \tan\alpha)$ and B $(a \sec\beta, b \tan\beta)$, then find the eq. of chord.

Sol: $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{Y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

Eq. of chord:

$$\Rightarrow (Y - b\tan\alpha) = \frac{b}{a}\left(\frac{\tan\beta-\tan\alpha}{\sec\beta-\sec\alpha}\right)(x - a\sec\alpha)$$

$$\Rightarrow \left(\frac{Y}{b} - \tan\alpha\right) = \frac{\sin(\beta-\alpha)}{\cos\alpha-\cos\beta}\left(\frac{x}{a}-\sec\alpha\right)$$

$$\Rightarrow \left(\frac{Y}{b} - \tan\alpha\right) = \frac{2\sin\left(\frac{\beta-\alpha}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right)\left(\frac{x}{a}-\sec\alpha\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{-\alpha+\beta}{2}\right)}$$

$$\Rightarrow \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) - \tan\alpha\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{x}{a}\cos\left(\frac{\beta-\alpha}{2}\right) - \sec\alpha\cos\left(\frac{\beta-\alpha}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right)\sec\alpha - \tan\alpha\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \cos\left(\alpha - \frac{(\alpha+\beta)}{2}\right)\sec\alpha - \frac{\sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\alpha} = \frac{\left\{\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right) - \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)\right\}}{\cos\alpha}$$

$$\Rightarrow \quad = \cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \boxed{\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)}$$

Ques: If a chord joining $A(\alpha)$ and $B(\beta)$ passes through the foci of hyperbola, then find the value of $\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}$

Sol: $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

$S \equiv (ae, 0)$

$$\Rightarrow \frac{ae}{a}\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow e = \cos\left(\frac{\alpha+\beta}{2}\right) \Big/ \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \frac{\pm e+1}{\pm e-1} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \quad \frac{\pm e+1}{\pm e-1} = \frac{2\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}}{2\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}} = \frac{1}{\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}}$$

$$\Rightarrow \quad -\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2} = \frac{\pm e-1}{\pm e+1}$$

$$\Rightarrow \quad \boxed{\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2} = \frac{1\mp e}{1\pm e}}$$

DIRECTOR CIRCLE

Let eq^n of tangent $y = mx \pm \sqrt{a^2m^2-b^2}$

$K = mh \pm \sqrt{a^2m^2-b^2}$

$(K-mh)^2 = a^2m^2-b^2$

$K^2+m^2h^2-2Kmh = a^2m^2-b^2$

$m^2(h^2-a^2)-2Kmh+K^2+b^2=0$

$m_1m_2 = -1 \qquad \frac{K^2+b^2}{h^2-a^2} = -1$

$K^2+b^2 = -(h^2-a^2)$

$K^2+h^2 = a^2-b^2$

$\boxed{x^2+y^2 = a^2-b^2}$ ⇒ Director circle

$a>b$	director circle	$1<e<\sqrt{2}$
$a=b$	It will be a pt. circle	$e=\sqrt{2}$
$a<b$	No director circle	$e>\sqrt{2}$

$$\boxed{e = \sqrt{1+\frac{b^2}{a^2}}}$$

CONJUGATE HYPERBOLA

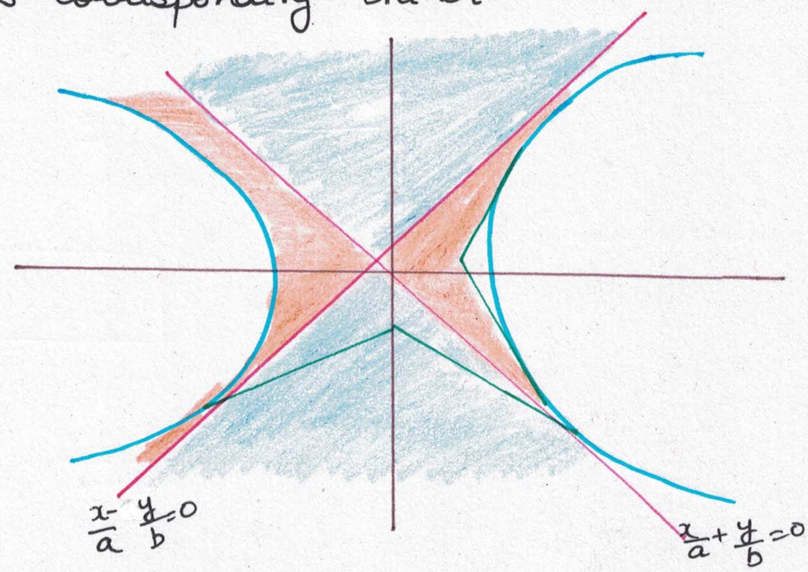
Ques: Find the eq. of a tangent to the hyperbola $\frac{x^2}{b^2}-\frac{y^2}{a^2}=-1$.

Sol: $y = mx+c$

$$\frac{(mx+c)^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$b^2m^2x^2 + b^2c^2 + 2mx\,cb^2 - a^2x^2 = a^2b^2$$

If the sign of output is +ve, then pt. P lies in orange region (above one assymptote and below the other). In this case, we can draw 2 tangents to its corresponding branch.



(I) If in violet region:

$P(x_1, y_1)$

$\frac{y}{b} - \frac{x}{a} = 0$ $\quad$ $\frac{y}{b} + \frac{x}{a} = 0$

$$\frac{y_1}{b} - \frac{x_1}{a} > 0$$

$$\frac{y_1}{b} + \frac{x_1}{a} > 0$$

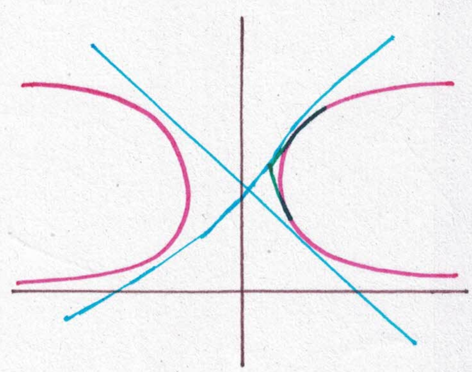
$$\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2} > 0 \quad \text{or} \quad \boxed{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 0}$$

(II) If in orange region: $\boxed{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} > 0}$

Ques: Consider the eq. of hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. If a pt. $P(2, \beta)$ lies outside the hyperbola, then find the interval of β so that the tangents drawn from this pt. touch only one branch.

Sol:

$$\frac{x_1^2}{16} - \frac{y_1^2}{9} > 0 \quad — (2, \beta)$$



$$\frac{4}{16} - \frac{\beta^2}{9} > 0$$

$$\frac{9 - 4\beta^2}{36} > 0$$

$$\frac{4\beta^2 - 9}{36} < 0$$

$$(2\beta + 3)(2\beta - 3) < 0$$

$$\boxed{\beta = -\frac{3}{2}, \frac{3}{2}}$$

Ques: Find the equation of asymptote of parabola, $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$. Also find its conjugate hyperbola.

Sol: First we'll find centre of hyperbola by partial differentiation.

$$f = 3x^2 + 10xy + 8y^2 + 14x + 22y + 7$$

$$\frac{\partial f}{\partial x} = 6x + 10y + 0 + 14 + 0 + 0$$

$$= 3x + 5y + 7 = 0$$

$$\frac{\partial f}{\partial y} = 0 + 10x + 16y + 0 + 22$$

$$= 5x + 8y + 11 = 0$$

Solve $3x + 5y + 7 = 0$ and $5x + 8y + 11 = 0$

$$\begin{array}{r} 15x + 25y + 35 = 0 \\ -\ 15x + 24y + 33 = 0 \\ \hline y + 2 = 0 \end{array} \Rightarrow y = -2$$

$$3(x) - 10 + 7 = 0$$

$$3x = 3 \quad , \quad x \Rightarrow 1$$

$$C \equiv (1, -2)$$

The POA pass through $C(1,-2)$

$$3(1) + 10(1)(-2) + 8(-2)^2 + 14(1) + 22(-2) + \lambda = 0$$

$$3 - 20 + 32 + 14 - 44 + \lambda = 0$$

$$\lambda = 15$$

$\boxed{\text{POA} :-\ 3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0}$ For constant term

$$CH + H = 2POA$$

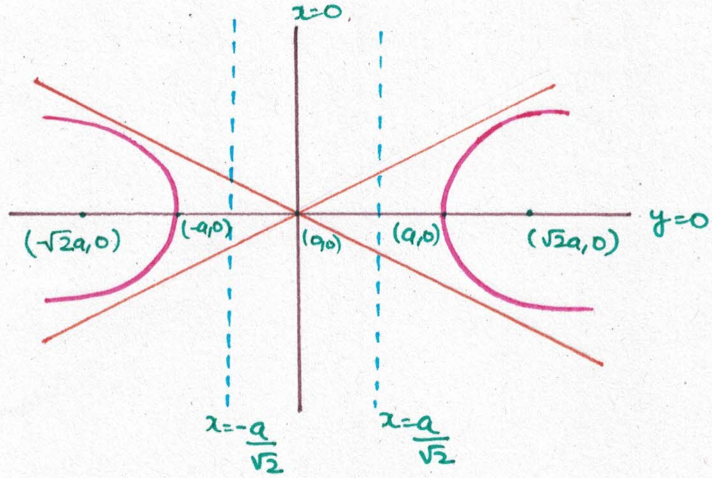
$$CH = 2POA - H \quad = 2(15) - 7$$

$$H = 23 = \cancel{H}$$

$\boxed{\text{Conjugate Hyperbola} :-\ 3x^2 + 10xy + 8y^2 + 14x + 22y + 23 = 0}$

RECTANGULAR HYPERBOLA

In hyperbola, if $a = b$, then it is called Rectangular hyperbola, whose eccentricity is $\sqrt{2}$



$C = (0,0)$ $\quad$ $S = (\pm\sqrt{2}a, 0)$

$V = (\pm a, 0)$ $\quad$ TA :- $y = 0$

CA :- $x = 0$ $\quad$ Directrix :- $x = \pm\frac{a}{\sqrt{2}}$

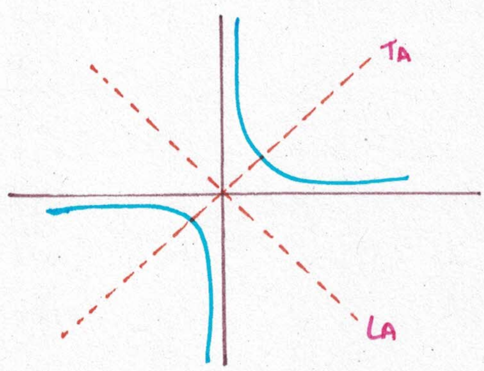
LOLR $= 2a$ $\quad$ LOTA $= 2a$ $\quad$ LOCA $= 2a$

End pts. of TA $= (\pm a, 0)$ $\quad$ End pts. of CA $= (0, \pm a)$

End pts. of LR $= (\pm\sqrt{2}a, \pm a)$ $\quad$ $e = \sqrt{2}$

LR $=$ $\quad$ $x = \pm a\sqrt{2}$

ROTATED HYPERBOLA



	x	y
X	$\cos\theta$	$\sin\theta$
Y	$-\sin\theta$	$\cos\theta$

$\theta = 45^\circ$ (anticlockwise)

If we rotate the co-ordinate axis by a angle 45° in anticlockwise direction, then the rectangular hyperbola will look like as in above diagram. → Rotated Rectangular Hyperbola

$$X = x\cos\theta + y\sin\theta$$
$$Y = -x\sin\theta + y\cos\theta$$

$$X = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \qquad Y = \frac{-x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

$$X^2 - Y^2 = a^2$$

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{x-y}{\sqrt{2}}\right)^2 = a^2$$

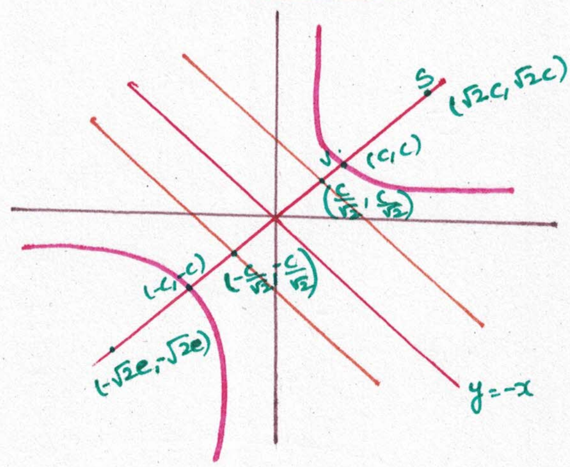
$$x^2+y^2+2xy-x^2-y^2+2xy = 2a^2$$

$$4xy = 2a^2$$

$$xy = \frac{a^2}{2}$$

Let $a^2/2 = c^2 \Rightarrow \boxed{c = \frac{a}{\sqrt{2}}}$

Rectangular Hyperbola :- $\boxed{xy = c^2}$



$c = a/\sqrt{2}$

Vertex: $\dfrac{x-0}{1/\sqrt{2}} = \dfrac{y-0}{1/\sqrt{2}} = \pm a$

$x = \frac{a}{\sqrt{2}} \qquad y = \frac{a}{\sqrt{2}} \qquad (c,c)$

$V = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and $\left(\frac{-a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

$(c,c) \qquad (-c,-c)$

Focus: $\dfrac{x-0}{1/\sqrt{2}} = \dfrac{y}{1/\sqrt{2}} = \pm\sqrt{2}a$

$x = a \qquad y = a \qquad\qquad x=-a \qquad y=-a$

$S(\pm\sqrt{2}c, \pm\sqrt{2}c)$

Directrix: $x+y-\sqrt{2}c = 0$

$x+y+\sqrt{2}c = 0$

LR : $x+y-2\sqrt{2}c = 0$

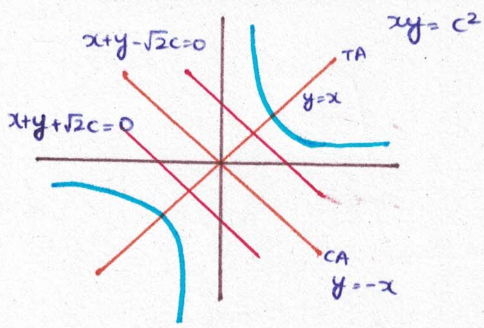
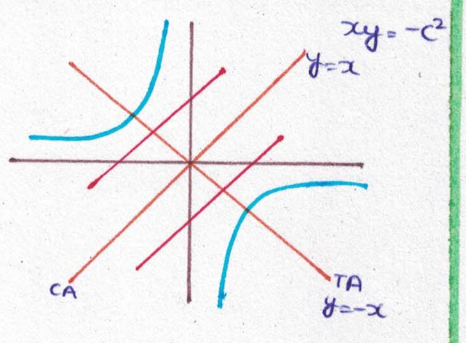
$x+y+2\sqrt{2}c = 0$

directrix :- $x = \pm \frac{a}{2}$ $\quad y = \pm \frac{a}{2}$

Directrix passes through $\left(\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$

$S = (\sqrt{2}c, \sqrt{2}c)$

$V = (c, c)$

		
Center	$(0,0)$	$(0,0)$
Vertex	(c,c) $(-c,-c)$	$(c,-c)$ $(-c,c)$
Focii	$(\sqrt{2}c, \sqrt{2}c)$ $(-\sqrt{2}c, -\sqrt{2}c)$	$(-\sqrt{2}c, \sqrt{2}c)$ $(\sqrt{2}c, -\sqrt{2}c)$
End pt. of CA	$(c,-c)$ $(-c,c)$	(c,c) $(-c,-c)$
TA	$y = x$	$y = -x$
CA	$y = -x$	$y = x$
Directrix	$x+y-\sqrt{2}c = 0$ $x+y+\sqrt{2}c = 0$	$x-y-\sqrt{2}c=0$ $x-y-\sqrt{2}c=0$
LR	$x+y+2\sqrt{2}c=0$ $x+y-2\sqrt{2}c=0$	$x-y+2\sqrt{2}c=0$ $x-y-2\sqrt{2}c=0$
LOTA	$2\sqrt{2}c$	$2\sqrt{2}c$
LOLR	$2\sqrt{2}c$	$2\sqrt{2}c$
LOCA	$2\sqrt{2}c$	$2\sqrt{2}c$
	$e = \sqrt{2}$	$e = \sqrt{2}$

POINTS TO REMEMBER

1. In the rotated rectangular hyperbola, the co-ordinate axis are asymptotes

2. The parametric co-ordinates of rotated rectangular hyperbola will be given as- $\left(\frac{c}{t}, ct\right)$

$xy = c^2$ $\quad \frac{x}{c} = \frac{c}{y} = t$ $\quad x = \frac{c}{t}, \; y = ct$

3. Every rectangular hyperbola, whether rotated or not, is also called equilateral hyperbola

4. The equation of a tangent to $xy = c^2$ is given as $T=0$

$$xy = c^2$$
$$2xy = 2c^2$$
$$xy + xy = 2c^2$$
$$\boxed{xy_1 + x_1 y = 2c^2} \quad \text{Slope Form}$$
$$x \cdot ct + \frac{c}{t} \cdot y = 2c^2$$
$$\boxed{xt + \frac{y}{t} = 2c} \quad \text{Parametric Form}$$

Ques: If the 2pts. $A(t_1)$ and $B(t_2)$ are joined to form a chord of the hyperbola $xy = c^2$. Find eq. of chord

Sol: $A = \left(\frac{c}{t_1}, ct_1\right)$ $\quad B = \left(\frac{c}{t_2}, ct_2\right)$

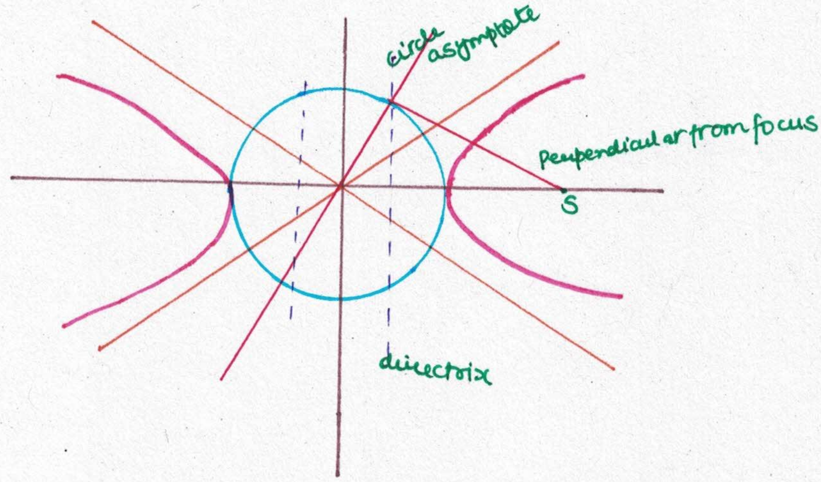
$$\frac{y - ct_2}{x - \frac{c}{t_2}} = \frac{c(t_2 - t_1)}{c\left(\frac{1}{t_2} - \frac{1}{t_1}\right)}$$
$$\frac{(y - ct_2)t_2}{xt_2 - c} = \frac{-1(t_1 - t_2)(t_1 t_2)}{(t_1 - t_2)}$$
$$y - ct_2 = -t_1(xt_2 - c)$$
$$y - ct_2 = -xt_1t_2 + ct_1$$
$$\boxed{xt_1t_2 + y = c(t_1 + t_2)}$$

PROPERTIES OF HYPERBOLA

1. The foot of the perpendicular drawn from the focus to any tangent of hyperbola, will meet an auxillary circle.

2. The product of the perpendicular drawn from focus of any tangent will be equal to b^2

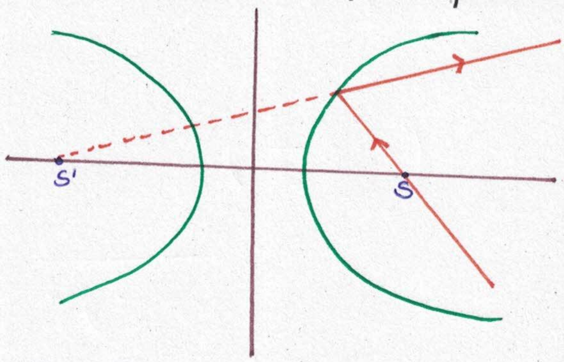
3. The foot of the perpendicular drawn from any focus to asymptotes are the common pts of auxillary circle and directrix.



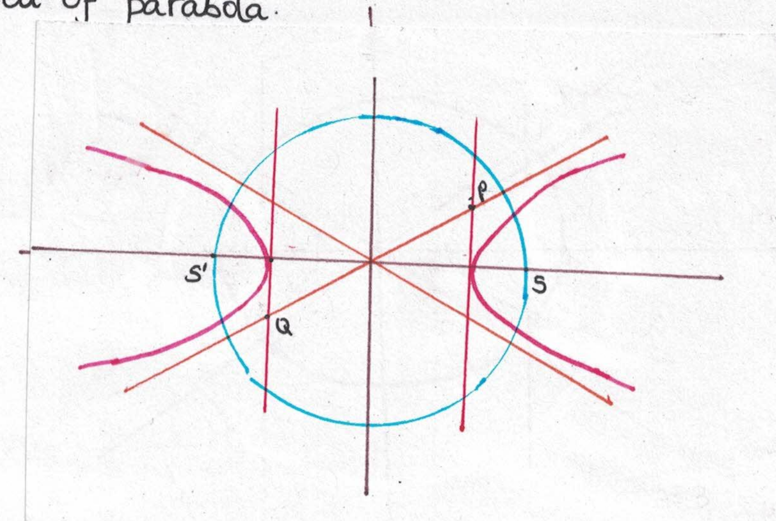
4 The portion of the tangent between point of contact and directrix Substends 90° at the corresponding focus.

5 Tangent and normal at any pt. is the bisector of angle between focal length

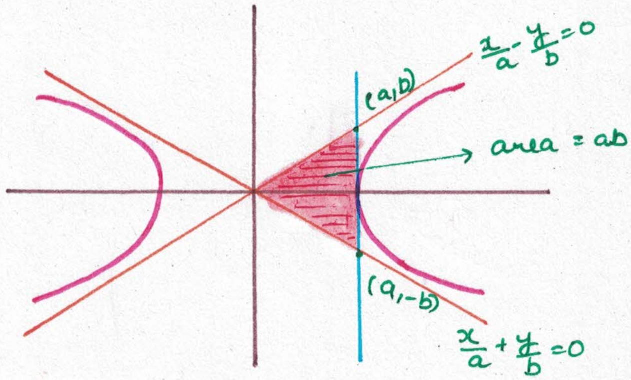
6 If a ray which seems to pass through one focus of the hyperbola after reflection will appear to pass from other focus



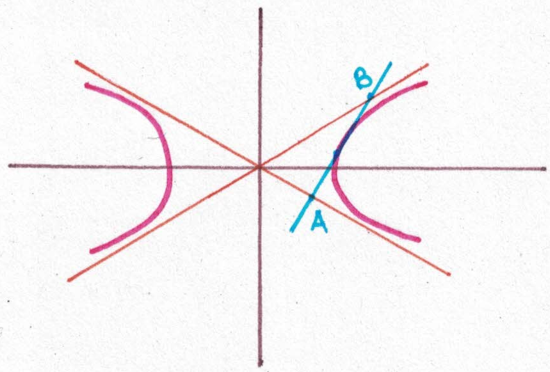
7 If P and Q are the POI of asymptotes and TAV, and the circle is made by taking PQ as diameter, then the circle will always pass through focii of parabola.



8 The Δ formed by 2 asymptotes and 1 tangent of hyperbola will always have constant area = ab



9 The portion of the tangent intercepted b/w asymptotes will have the mid pt. as their point of contact.



10 If the angle b/w the asymptotes is given as 2θ for $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the eccentricity of the hyperbola will be $\sec\theta$

$$\tan 2\theta = \left|\frac{\frac{b}{a}+\frac{b}{a}}{1-b^2/a^2}\right|$$

$$\tan 2\theta = \frac{2b/a}{(a^2-b^2)/a^2}$$

$$\tan 2\theta = \left|\frac{2ab}{a^2-b^2}\right| \qquad \cos 2\theta = \frac{a^2-b^2}{(a^2+b^2)^2}$$

$$e^2 = \frac{a^2+b^2}{a^2}$$

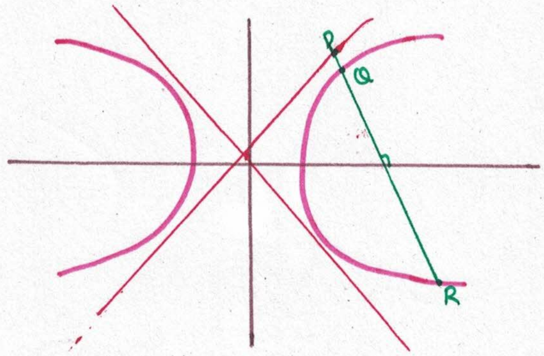
11 If α, β, γ and δ are eccentric angles of 4 co-normal pts. then-

$\alpha+\beta+\gamma+\delta =$ odd multiple of π

12 If α, β, γ and δ are eccentric angles of 4 concyclic pts. of hyperbola, then-

$\alpha+\beta+\gamma+\delta =$ Even multiple of π

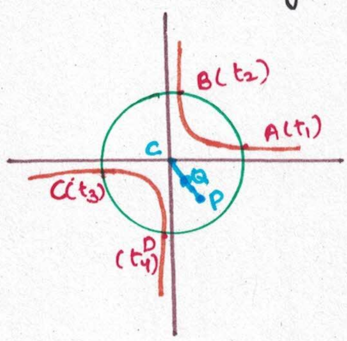
13 From any pt. on asymptotes, a straight line be drawn perpendicular to transverse axis which cuts one branch of hyperbola at Q and R, then $PQ \cdot PR = b^2$



$PQ \cdot PR = b^2$

PROPERTIES OF RECTANGULAR HYPERBOLA

14 The mean position of POI of circle and rectangular hyperbola bisects the line joining centre of circle and hyperbola.



$$Q = \left[\frac{c\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right)}{4}, \frac{c(t_1+t_2+t_3+t_4)}{4}\right]$$

15 If the vertex of the Δ are lying on the rectangular hyperbola, then its orthocenter will also lie on the rectangular hyperbola

16 If the POI of a circle and rectangular hyperbola having the parametric coordinates as t_1, t_2, t_3, t_4, then–

$$\boxed{t_1 \cdot t_2 \cdot t_3 \cdot t_4 = 1}$$

17 If t_1, t_2, t_3, t_4 are the parametric co-ordinates of 4 co-normal pts. then–

$$\boxed{t_1 \cdot t_2 \cdot t_3 \cdot t_4 = -1}$$